

On a Theorem Regarding Lattice Pentagons

Cosmin Pohoța

In this paper we call the points from the Euclidean plane \mathbb{R}^2 belonging to $\mathbb{Z} \times \mathbb{Z}$ *lattice points*. In addition, we will call a pentagon having all its vertices at lattice points a *lattice pentagon*.

In 1980, Arkinstall [2] proved the following result, known as the *the lattice pentagon theorem*:

Theorem 1. *A convex lattice pentagon must contain a lattice point in its interior.*

Twenty years later, a refinement of this result appeared as a proposed problem in the Russian Mathematical Olympiad.

Theorem 2. *If \mathcal{P} is a convex pentagon having the vertices on lattice points, the smaller pentagon $S(\mathcal{P})$ determined by its diagonals contains, in its interior or on its boundary, a lattice point.*

Several approaches using Pick's celebrated formula (see, for example, [3]) lead to proofs of this beautiful theorem. The most frequent one, in this spirit, is the one from [1], which combines Pick's theorem with an extrema argument. In the following, we give a proof which avoids Pick's formula, based on Arkinstall's original idea from [2].

Proof of Theorem 2. Let A, B, C, D, E be the vertices of \mathcal{P} . Since a lattice point can only be one of the four types: $(2\mathbb{Z}+1, 2\mathbb{Z}+1)$, $(2\mathbb{Z}, 2\mathbb{Z})$, $(2\mathbb{Z}+1, 2\mathbb{Z})$, $(2\mathbb{Z}, 2\mathbb{Z}+1)$, by the pigeonhole principle, two of the vertices of $ABCDE$ will have the same form, and therefore the midpoint M of the segment determined by these two points is also a lattice point. We now distinguish the two cases depending on whether or not these two vertices are adjacent:

1. The point M lies on a diagonal of the pentagon $ABCDE$.

Without loss of generality, assume that M is the midpoint of segment AC . If it lies on the boundary of $S(ABCDE)$ we are done. If it does not, one of the two pentagons $ABMDE$ and $MBCDE$ is convex, say $MBCDE$. This convex pentagon is a lattice pentagon contained in the initial $ABCDE$. We observe that $S(MBCDE)$ is included in $S(ABCDE)$, as well. Thus, by inducting on the number of lattice points contained in the initial pentagon \mathcal{P} , $S(MBCDE)$ will contain a lattice point, and then so will $S(ABCDE)$, as desired.

2. The point M lies on an edge of the pentagon $ABCDE$.

Without loss of generality, assume that M is the midpoint of AB . The pentagon $MBCDE$ is convex. Although $S(MBCDE)$ is not included in $S(ABCDE)$ anymore, we see that $S(MBCDE)$ is included in the union of $S(ABCDE)$ with the triangle \mathbf{T} bounded by the three lines BD , BE , AC . According to a similar inductive argument, $S(MBCDE)$ contains a lattice point M' . If M' lies in the interior of $S(ABCDE)$ we are done. If it does not, it lies inside the triangle \mathbf{T} , and since the pentagon $AM'CDE$ is convex, with its small pentagon $S(AM'CDE)$ included in $S(ABCDE)$, by applying the same inductive argument, we conclude that $S(AM'CDE)$ contains a lattice point, which lies inside $S(ABCDE)$ as well. This proves Theorem 2.

References

- [1] T. Andreescu, Z. Feng, G. Lee, *Mathematical Olympiads, 2000-2001: Problems and Solutions from Around the World*, MAA, 2003.
- [2] J. R. Arkininstall, Minimal Requirements for Minkowski's Theorem in the Plane I, *Bulletin of the Australian Mathematical Society.*, 22(1980)259-274.
- [3] G. Pick, Geometrisches zur Zahlentheorie, *Sitzber. Lotos* (Prague) 19, 311-319, 1899.

Cosmin Pohoățã

Tudor Vianu National College
Bucharest, Romania RO-010014
pohoata_cosmin2000@yahoo.com